Factorization

4

When doctors prescribe medication, they give patients instructions as to how much to take and how often it should be taken. The amount of medication in your body varies with time.

Suppose the equation $M(t) = 3.5t^3 - 100t^2 + 350t$ models the number of milligrams of a certain medication in the bloodstream t hours after it has been taken Clearly, M(t) represents a polynomial (cubic) in variable t. The doctor can use the roots of this equation to determine how often the patient should take the medication to maintain a certain concentration in the body.

"Polynomials are used in the real world, people in all sorts of professions use polynomials everyday. Although polynomials offer limited information, they can be used in more sophisticated way to retrieve more data."

4.1 Polynomial in one variable

An algebraic expression of the form

$p(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1x^1 + a_0x^0$, where

(i) $a_n \neq 0$

(ii) $a_0, a_1, a_2, ... a_n$ are real numbers

(iii) power of x is a positive integer, is called a polynomial in one variable. Hence, $a_n, a_{n-1}, a_{n-2}, ..., a_0$ are coefficients of $x^n, x^{n-1}, ..., x^0$ respectively and $a_nx^n, a_{n-1}x^{n-1}, a_{n-2}x^{n-2}, ...$ are terms of the polynomial. Here the term a_nx^n is called the ***leading term*** and its coefficient a_n, the ***leading coefficient.***

(iv) If all the coefficients $a_0, a_1, a_2, ..., a_{n-1}, a_n$ are zero, then f(x) is called a zero polynomial; it is denoted by the symbol 0. The degree of the zero polynomial is never defined.

(v) The degree of a polynomial is zero if and only if it is a non-zero constant polynomial. Thus, if deg f(x) = 0, then f(x) is a (non-zero) constant polynomial; it is usually denoted by c i.e. f(x) = c, $c \neq 0$.

THE SPOT LIGHT ✓

Are all algebraic expressions polynomials ?

4.2 Degree of polynomials

Degree of the polynomial in one variable is the largest exponent of the variable. For example, the degree of the polynomial $3x^7 - 4x^6 + x + 9$ is 7 and the degree of the polynomial $5x^6 - 4x^2 - 6$ is 6.

Polynomials classified by degree

Degree	Name	General form	Example
$-\infty$ (undefined)	Zero polynomial	0	0
0	(Non-zero) constant polynomial	$a; (a \neq 0)$	1
1	Linear polynomial	$ax + b; (a \neq 0)$	$x + 1$
2	Quadratic polynomial	$ax^2 + bx + c; (a \neq 0)$	$x^2 + 1$
3	Cubic polynomial	$ax^3 + bx^2 + cx + d; (a \neq 0)$	$x^3 + 1$

Usually, a polynomial of degree n, for n greater than 3, is called a polynomial of degree n, although the phrases quartic polynomial and quintic polynomial are sometimes used.

THE SPOT LIGHT

A non-zero constant is a polynomial of degree zero, but the degree of zero polynomial is not defined.

Some special types of polynomials

Monomials	: Polynomials having only one term are called monomials. E.g. 2, 2x, $7y^5$, $12t^7$ etc.
Binomials	: Polynomials having exactly two terms are called binomials. E.g. $p(x) = 2x + 1$, $r(y) = 2y^7 + 5y^6$. etc.
Trinomials	: Polynomials having exactly three terms are called trinomials. E.g. $p(x) = 2x^2 + x + 6$, $q(y) = 9y^6 + 4y^2 + 1$ etc.

BUILDING CONCEPTS 4.1

Write whether the following statements are True or False. Justify your answer.

(i) $\dfrac{1}{\sqrt{5}}x^{1/2} + 1$ is a polynomial

(ii) $\dfrac{6\sqrt{x}+x^{3/2}}{\sqrt{x}}$ is a polynomial, $x \neq 0$

Explanation

(i) False, because the exponent of the variable is not a whole number.

(ii) True, because $\dfrac{6\sqrt{x}+x^{3/2}}{\sqrt{x}} = 6 + x$, which is a polynomial.

BUILDING CONCEPTS 4.2

For the polynomial $\dfrac{x^3+2x+1}{5} - \dfrac{7}{2}x^2 - x^6$, write

(i) The degree of the polynomial

(ii) The coefficient of x^3

(iii) The coefficient of x^6

(iv) The constant term

Explanation

Polynomial $\dfrac{x^3+2x+1}{5} - \dfrac{7}{2}x^2 - x^6$ can be written as

$$\frac{x^3}{5} + \frac{2}{5}x + \frac{1}{5} - \frac{7}{2}x^2 - x^6$$

or $$-x^6 + \frac{x^3}{5} - \frac{7}{2}x^2 + \frac{2}{5}x + \frac{1}{5}$$

or $$(-1)x^6 + (0)x^5 + (0)x^4 + \left(\frac{1}{5}\right)x^3 + \left(\frac{-7}{2}\right)x^2 + \left(\frac{2}{5}\right)x + \frac{1}{5}$$

So, (i) The degree of the polynomial = 6

(ii) The coefficient of $x^3 = \left(\dfrac{1}{5}\right)$

(iii) The coefficient of $x^6 = -1$

(iv) The constant term $= \dfrac{1}{5}$

THE SPOT LIGHT

If the sum of the co-efficients of a polynomial is zero, then (x – 1) is a factor of the polynomial.

BUILDING CONCEPTS 4.3

Which of the following expressions are polynomials? Justify your answer.

(i) 8

(ii) $\sqrt{3}x^2 - 2x$

(iii) $1 - \sqrt{5x}$

(iv) $\frac{1}{5x^{-2}} + 5x + 7$

(v) $\frac{(x-2)(x-4)}{x}$

(vi) $\frac{1}{x+1}$

(vii) $\frac{1}{7}a^3 - \frac{2}{\sqrt{3}}a^2 + 4a - 7$

(viii) $\frac{1}{2x}$

Explanation

(i), (ii), (iv), (vii) are polynomials because the exponent of the variable after simplification in each of these is a whole number.

THE SPOT LIGHT ✓

A polynomial in x is said to be a polynomial in standard form, if the powers of x are either in ascending order or in descending order.

BUILDING CONCEPTS 4.4

Classify the following as linear, quadratic & cubic polynomials.

(i) $y + y^3 + 4$

(ii) $5x - \frac{3}{2}$

(iii) $x^2 - 2x + 3$

(iv) $3xyz$

Explanation

(i) It is a cubic polynomial because its degree is 3.

(ii) It is a linear polynomial because its degree is 1.

(iii) It is quadratic polynomial because its degree is 2.

(iv) It is a cubic polynomial because its degree is 1 + 1 + 1 = 3.

4.3 Division algorithm for polynomials

If a polynomial f(x) is divided by a non-zero polynomial g(x) then there exist unique polynomials q(x) and r(x) such that f(x) = g(x) q(x) + r(x) where either r(x) = 0 or deg r(x) < deg g(x). Here, dividend = f(x), divisor = g(x), quotient = q(x) and remainder = r(x).

Remarks

Let f(x) be any polynomial and it be divided by a non-zero polynomial g(x) and r(x) be the remainder.

- If g(x) is a quadratic polynomial then r(x) is of the form ax + b, where a, b may be zero.
- If g(x) is a linear polynomial then r(x) is a constant polynomial i.e. r(x) = c, where c may be zero.

Example : (Fig. 1)
Let us divide $2x^3 - 7x^2 + 5x - 9$ by $2x - 3$

Here, quotinet = $q(x) = x^2 - 2x - \frac{1}{2}$

and remainder = $r(x) = -\frac{21}{2}$.

$$
\begin{array}{rl}
 & x^2 - 2x - \frac{1}{2} \\
2x - 3 \,\big) & \overline{2x^3 - 7x^2 + 5x - 9} \\
 & 2x^3 - 3x^2 \\
 & \underline{-\quad\;\; +\qquad\qquad\quad} \\
 & \quad -4x^2 + 5x \\
 & \quad -4x^2 + 6x \\
 & \quad \underline{+\qquad -\qquad} \\
 & \qquad\quad -x - 9 \\
 & \qquad\quad -x + \frac{3}{2} \\
 & \qquad\quad \underline{+\quad -\;\;} \\
 & \qquad\qquad -\frac{21}{2}
\end{array}
$$

Fig. 1

4.4 Zeros / Roots of a polynomial / equation

Consider a polynomial $f(x) = 3x^2 - 4x + 2$. If we replace x by 3 everywhere in the above expression, we get

$f(3) = 3 \times (3)^2 - 4 \times 3 + 2 = 27 - 12 + 2 = 17$

We can say that the value of the polynomial f(x) at x = 3 is 17.

Similarly the value of polynomial $f(x) = 3x^2 - 4x + 2$

at $x = -2$ is $f(-2) = 3(-2)^2 - 4 \times (-2) + 2 = 12 + 8 + 2 = 22$

at $x = 0$ is $f(0) = 3(0)^2 - 4(0) + 2 = 0 - 0 + 2 = 2$

at $x = \frac{1}{2}$ is $f\left(\frac{1}{2}\right) = 3 \times \left(\frac{1}{2}\right)^2 - 4 \times \left(\frac{1}{2}\right) + 2 = \frac{3}{4} - 2 + 2 = \frac{3}{4}$

In general, we can say f(α), if the value of the polynomial f(x) at x = α, where α is a real number.

A real number α is zero of a polynomial f(x), if the value of the polynomial f(x) is zero at x = α i.e. f(α) = 0.

OR

The value of the variable x, for which the polynomial f(x) becomes zero is called zero of the polynomial.

THE SPOT LIGHT

A non-zero constant polynomial has no zero.

E.g. : Consider, a polynomial $p(x) = x^2 - 5x + 6$; replace x by 2 and 3.

$p(2) = (2)^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$,

$p(3) = (3)^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

∴ 2 and 3 are the zeros of the polynomial p(x).

Roots of a polynomial equation

An expression f(x) = 0 is called a polynomial equation, if f(x) is a polynomial of degree $n \geq 1$.

A real number α is a root of a polynomial f(x) = 0, if f(α) = 0 i.e. α is a zero of the polynomial f(x).

E.g. Consider the polynomial $f(x) = 3x - 2$, then $3x - 2 = 0$ is the corresponding polynomial equation.

Here, $f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right) - 2 = 0$

i.e. $\frac{2}{3}$ is a zero of the polynomial $f(x) = 3x - 2$

or $\frac{2}{3}$ is a root of the polynomial equation $3x - 2 = 0$

NUMERICAL ABILITY 4.1

Find q(0), q(1) and q(2) for each of the following polynomials :

(i) $q(x) = x^2 + 3x$ (ii) $q(y) = 2 + y + 2y^2 - 5y^3$

Solution

(i) $q(x) = x^2 + 3x$

$\therefore$ $q(0) = (0)^2 + 3 \times 0 = 0$

$q(1) = (1)^2 + 3 \times 1 = 4$

$q(2) = (2)^2 + 3 \times 2 = 4 + 6 = 10$

(ii) $q(y) = 2 + y + 2y^2 - 5y^3$

$\therefore$ $q(0) = 2 + 0 + 2(0)^2 - 5(0)^3 = 2$

$q(1) = 2 + 1 + 2(1)^2 - 5(1)^3 = 2 + 1 + 2 - 5 = 0$

and $q(2) = 2 + 2 + 2(2)^2 - 5(2)^3 = 2 + 2 + 8 - 40 = -28$

NUMERICAL ABILITY 4.2

Find $q(a + 1) - 2q(a)$ if $q(x) = x^2 + 3x + 4$.

Solution

To evaluate $q(a + 1)$, replace x in q(x) with a + 1.

$q(x) = x^2 + 3x + 4$

$q(a + 1) = (a + 1)^2 + 3(a + 1) + 4$

$= a^2 + 2a + 1 + 3a + 3 + 4 = a^2 + 5a + 8$

To evaluate 2q(a), replace x with a in q(x), then multiply the expression by 2.

$q(x) = x^2 + 3x + 4$

$2q(a) = 2(a^2 + 3a + 4) = 2a^2 + 6a + 8$

Now, evaluate $q(a + 1) - 2q(a)$

$q(a + 1) - 2q(a) = a^2 + 5a + 8 - (2a^2 + 6a + 8)$

$= a^2 + 5a + 8 - 2a^2 - 6a - 8 = -a^2 - a$

BUILDING CONCEPTS 4.5

BOATING A motor boat traveling against waves accelerates from a resting position. Suppose the speed of the boat in metres per second is given by the function $f(t) = -0.04t^4 + 0.8t^3 + 0.5t^2 - t$, where t is the time in seconds.

(i) Find the speed of the boat at 1, 2 and 3 seconds.

(ii) It takes 6 seconds for the boat to travel between two buoys while it is accelerating. Find f(6) and explain what it means.

Explanation

(i) Speed of the boat at 1 second

$= f(1) = -0.04(1)^4 + 0.8(1)^3 + 0.5(1)^2 - 1$

$= -0.04 + 0.8 + 0.5 - 1 = -1.04 + 1.3 = 0.26$ m/sec.

Speed of the boat at 2 second

$= f(2) = -0.04(2)^4 + 0.8(2)^3 + 0.5(2)^2 - 2$

$= -0.04(16) + 0.8(8) + 0.5(4) - 2$

$= -0.64 + 6.4 + 2.0 - 2 = 5.76$ m/sec.

Fig.1

THE SPOT LIGHT

Every linear polynomial has one and only one real zero.

Speed of the boat at 3 second

$= f(3) = -0.04\,(3)^4 + 0.8\,(3)^3 + 0.5\,(3)^2 - 3$

$= -0.04(81) + 0.8(27) + 0.5(9) - 3$

$= -3.24 + 21.6 + 4.5 - 3$

$= -6.24 + 26.1 = 22.86$ m/sec.

(ii) $f(6) = -0.04\,(6)^4 + 0.8\,(6)^3 + 0.5\,(6)^2 - 6$

$= -51.84 + 172.8 + 18 - 6$

$= -57.84 + 190.8 = 132.96$ m/sec.

This means, the speed of the boat at 6th second = 132.96 m/sec.

NUMERICAL ABILITY 4.3

Verify whether the indicated values of variables are zeros of the polynomials corresponding to them :

(i) $p(y) = 4y - 4\pi,\ y = 4, \pi$ (ii) $q(u) = (u + 1)(u + 2),\ u = -1, 2$

Solution

(i) $p(y) = 4y - 4\pi$

$p(4) = 4(4) - 4\pi = 16 - 4\pi \neq 0$

$p(\pi) = 4\pi - 4\pi = 0$

$\Rightarrow$ π is a zero and 4 is not a zero of the polynomial.

(ii) $q(u) = (u + 1)(u + 2)$

$q(-1) = (-1 + 1)(-1 + 2) = (0)(1) = 0$

$q(2) = (2 + 1)(2 + 2) = (3)(4) = 12 \neq 0$

$\Rightarrow$ -1 is a zero and 2 is not a zero of the polynomial.

THE SPOT LIGHT ✓

When a polynomial is divided by one of its binomial factors, the quotient obtained is called a depressed polynomial. Can you tell the difference between the degrees of the original polynomial and the depressed polynomial ?

NUMERICAL ABILITY 4.4

Find the zero of the polynomials given below :

(i) $p(x) = 3x + \pi$ (ii) $q(u) = (u + 3)(u - 4)$

Solution

(i) $p(x) = 3x + \pi$

It's corresponding polynomial equation is $3x + \pi = 0$

$\Rightarrow 3x = -\pi$ or $x = -\dfrac{\pi}{3}$

$\therefore -\dfrac{\pi}{3}$ is a zero of the polynomial.

(ii) $q(u) = (u + 3)(u - 4)$

It's corresponding polynomial equation is $(u + 3)(u - 4) = 0$

$\Rightarrow$ either $u + 3 = 0$ or $u - 4 = 0$

$\Rightarrow$ $u = -3$ or $u = 4$

$\therefore$ -3 and 4 are the zeros of the polynomial.

4.5 Remainder theorem

Statement : Let p(x) be a polynomial of degree ≥ 1 and 'a' is any real number. If p(x) is divided by $(x - a)$, then the remainder is p(a).

E.g. : Let p(x) be $x^3 - 7x^2 + 6x + 4$

Divide p(x) with $(x - 6)$ and to find the remainder, put $x = 6$ in p(x) i.e. p(6) will be the remainder.

∴ Required remainder be

$$\begin{aligned} p(6) &= (6)^3 - 7.6^2 + 6.6 + 4 \\ &= 216 - 252 + 36 + 4 \\ &= 256 - 252 = 4 \end{aligned}$$

$$\begin{array}{r|l|l} x-6 & x^3 - 7x^2 + 6x + 4 & x^2 - x \\ & \underline{\underset{-}{x^3} \underset{+}{-6x^2}} & \\ & -x^2 + 6x + 4 & \\ & \underline{\underset{+}{-x^2} \underset{-}{+6x}} & \\ & \text{Remainder} = 4 & \end{array}$$

Thus, p(a) is remainder on dividing p(x) by (x – a).

Remark (i) p(–a) is remainder on dividing p(x) by (x + a)

[∵ $x + a = 0 \Rightarrow x = -a$]

(ii) $p\left(\frac{b}{a}\right)$ is remainder on dividing p(x) by (ax – b)

[∵ $ax - b = 0 \Rightarrow x = b/a$]

(iii) $p\left(\frac{-b}{a}\right)$ is remainder on dividing p(x) by (ax + b)

[∵ $ax + b = 0 \Rightarrow x = -b/a$]

(iv) $p\left(\frac{b}{a}\right)$ is remainder on dividing p(x) by (b – ax)

[∵ $b - ax = 0 \Rightarrow x = b/a$]

THE SPOT LIGHT

A quadratic polynomial $ax^2 + bx + c$, $a \neq 0$ can have at most two real zeros. In some cases, it may not have any real zero.

NUMERICAL ABILITY 4.5

Find the remainder when

(i) $x^3 - ax^2 + 6x - a$ is divided by $x - a$

(ii) $2x^4 + x^3 - 2x^2 + x + 1$ by $2x - 1$

Solution

(i) Let $p(x) = x^3 - ax^2 + 6x - a$

zero of $x - a$ is a

$p(a) = a^3 - a(a)^2 + 6(a) - a = a^3 - a^3 + 6a - a = 5a$

So, by the Remainder theorem, remainder = 5a

(ii) Let $p(x) = 2x^4 + x^3 - 2x^2 + x + 1$

zero of $2x - 1$ is $1/2$

So,

$$p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right) + 1$$

$$= \frac{1}{8} + \frac{1}{8} - \frac{1}{2} + \frac{1}{2} + 1 = \frac{1}{4} + 1 = \frac{5}{4}$$

So, by the Remainder theorem, remainder = $\frac{5}{4}$

NUMERICAL ABILITY 4.6

If the polynomials $ax^3 + 4x^2 + 3x - 4$ and $x^3 - 4x + a$ leave the same remainder when divided by $(x - 3)$, find the value of a.

Solution

Let $p(x) = ax^3 + 4x^2 + 3x - 4$ and $q(x) = x^3 - 4x + a$ be the given polynomials. The remainders when p(x) and q(x) are divided by (x − 3) are p(3) and q(3) respectively.

By the given condition, we have p(3) = q(3)

$\Rightarrow \quad a \times (3)^3 + 4 \quad (3)^2 + 3 \times 3 - 4 = (3)^3 - 4 \quad 3 + a$

$\Rightarrow \quad 27a + 36 + 9 - 4 = 27 - 12 + a$

$\Rightarrow \quad 26a + 26 = 0$

$\Rightarrow \quad 26a = -26$

$\Rightarrow \quad a = -1$

4.6 Factor theorem

Statement : Let f(x) be a polynomial of degree ≥ 1 and a be any real constant such that f(a) = 0, then (x − a) is a factor of f(x). Conversely, if (x − a) is a factor of f(x), then f(a) = 0.

Proof : By Remainder theorem, if f(x) is divided by (x − a), the remainder will be f(a). Let q(x) be the quotient. Then, we can write,

$f(x) = (x - a) \times q(x) + f(a)$ (∵ Dividend = Divisor × Quotient + Remainder)

If f(a) = 0, then $f(x) = (x - a) \times q(x)$

Thus, (x − a) is a factor of q(x).

Converse Let (x − a) is a factor of f(x).

Then we have a polynomial q(x) such that $f(x) = (x - a) \times q(x)$

Replacing x by a, we get f(a) = 0.

Hence proved.

NUMERICAL ABILITY 4.7

Use the factor theorem to determine whether (x − 1) is a factor of

$f(x) = 2\sqrt{2}x^3 + 5\sqrt{2}x^2 - 7\sqrt{2}$

Solution

By using factor theorem, (x − 1) is a factor of f(x), only when f(1) = 0

$$f(1) = 2\sqrt{2}(1)^3 + 5\sqrt{2}(1)^2 - 7\sqrt{2}$$

$$= 2\sqrt{2} + 5\sqrt{2} - 7\sqrt{2} = 0$$

Hence, (x − 1) is a factor of f(x).

NUMERICAL ABILITY 4.8

For what value of k, (x − 1) is a factor of $p(x) = kx^2 - 3x + k$?

Solution

Here $p(x) = kx^2 - 3x + k$

∵ x − 1 is a factor of p(x)

∴ x − 1 = 0

$\therefore \quad x = 1$

$\therefore \quad p(1) = 0 \quad$ or $\quad k(1)^2 - 3(1) + k = 0$

$\Rightarrow \quad k - 3 + k = 0 \quad$ or $\quad 2k - 3 = 0 \quad \therefore \quad k = \frac{3}{2}$

4.7 Factorisation of polynomials

Factorisation of a quadratic polynomial by splitting the middle term

Let (px + q) and (rx + s) be the 2 linear factors of the quadratic polynomial $ax^2 + bx + c$, $(a \neq 0)$.

Then $ax^2 + bx + c = (px + q)(rx + s) = prx^2 + (ps + qr)x + qs$

On comparing the coefficients, we get a = pr; b = ps + qr and c = qs

Clearly b is the sum of two numbers ps and qr, whose product (ps qr) is equal to ac.

This means, for factorising a quadratic polynomial $ax^2 + bx + c$ into two linear factors, we need to write b as sum of two numbers whose product is ac.

NUMERICAL ABILITY 4.9

Factorise the quadratic polynomial $3x^2 + 7x + 2$ by splitting the middle term.

Solution

$p(x) = 3x^2 + 7x + 2$

The coefficient of the middle term is 7. Now, we find two numbers u and v such that u + v = 7 and u × v = 3 × 2 = 6. By inspection, we find u = 1 and v = 6. Then we have

$$\begin{aligned} 3x^2 + 7x + 2 &= 3x^2 + (1 + 6)x + 2 \qquad \{\text{By splitting the middle term}\} \\ &= 3x^2 + x + 6x + 2 \\ &= (3x^2 + x) + (6x + 2) \\ &= x(3x + 1) + 2(3x + 1) \\ &= (3x + 1)(x + 2) \end{aligned}$$

THE SPOT LIGHT

If the sum of the co-efficients of a polynomial is zero, then (x – 1) is a factor of the polynomial.

Factorisation of quadratic and cubic polynomials by factor theorem

In this method for factorising a polynomial into linear factors, obtain the constant term by making the coefficient of x^2 unity.

E.g. Consider a quadratic polynomial $ax^2 + bx + c$ $(a \neq 0)$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Let (x + p) and (x + q) be the two linear factors of $\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$.

Then, $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a(x + p)(x + q) = a[x^2 + (p + q)x + pq]$

On comparing the coefficients, we get $\frac{c}{a} = pq$

Clearly the two zeros (–p) and (–q) of the quadratic polynomial $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ are factors of the constant term $\frac{c}{a}$

This is the concept behind using the factor theorem for factorization.

Algorithm to factorise a quadratic or cubic polynomial

Step-1 : Obtain the constant term by making the coefficient of x^2 unity.

Step-2 : Take one of the factors of the constant term say a and replace x by it in the given example. If the polynomial reduces to zero, then $(x - a)$ is a factor of polynomial.

Step-3 : Similarly, obtain the other factors.

or

In case of quadratic polynomial, divide the given polynomial by $(x - a)$ to get the other linear factor.

In case of cubic polynomial, divide the given polynomial by $(x - a)$, to get the other factor (quadratic). Factorise the quadratic factor so obtained by the appropriate method to get the 2 linear factors.

THE SPOT LIGHT ✓

If the sum of the coefficients of odd powers in a polynomial is equal to same of coefficients of even powers, then $(x + 1)$ is a factor of the polynomial.

NUMERICAL ABILITY 4.10

Factorise $x^2 - 5x - 24$ by using the factor theorem.

Solution

$p(x) = x^2 - 5x - 24$.

Here, coefficient of the leading term is 1 and the constant term is – 24. A zero of the polynomial p(x) will be a factor of the number – 24. By inspection, we find that the number 8 is a divisor of – 24 and also we have

$$p(8) = (8)^2 - 5(8) - 24 = 64 - 40 - 24 = 0$$

i.e. 8 is a zero of the polynomial p(x).

Then $(x - 8)$ is a factor of the polynomial. We can express

$$\begin{aligned} x^2 - 5x - 24 &= (x^2 - 8x) + (3x - 24) \\ &= x(x - 8) + 3(x - 8) = (x - 8)(x + 3) \end{aligned}$$

We can also find the second factor of the polynomial by dividing $x^2 - 5x - 24$ by $(x - 8)$.

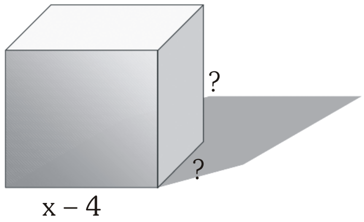

Fig.2

NUMERICAL ABILITY 4.11

Factorise $x^3 - 23x^2 + 142x - 120$.

Solution

$p(x) = x^3 - 23x^2 + 142x - 120$.

The coefficient of the leading term is 1 and the constant term is – 120. Factors of – 120 are many but we find a suitable factor of – 120 which is a zero of the polynomial.

By inspection, we find that

$$p(1) = (1)^3 - 23(1)^2 + 142(1) - 120 = 1 - 23 + 142 - 120 = 0$$

$\Rightarrow$ $(x - 1)$ is a factor of the polynomial.

Now, we can express the given polynomial as below :

$$\begin{aligned} x^3 - 23x^2 + 142x - 120 &= (x^3 - x^2) + (-22x^2 + 22x) + (120x - 120) \\ &= x^2(x - 1) - 22x(x - 1) + 120(x - 1) \\ &= (x - 1)(x^2 - 22x + 120) \\ &= (x - 1)\{x^2 + (-12 - 10)x + 120\} \\ &= (x - 1)\{(x^2 - 12x) + (-10x + 120)\} \\ &= (x - 1)\{x(x - 12) - 10(x - 12)\} \\ &= (x - 1)\{(x - 12)(x - 10)\} \\ &= (x - 1)(x - 10)(x - 12) \end{aligned}$$

BUILDING CONCEPTS 4.6

The volume of the rectangular prism is given by v (x) = $x^3 + 3x^2 - 36x + 32$. Find the missing measures.

Explanation

The volume of a rectangular prism is $\ell \times w \times h$. One measure is $x - 4$, so $x - 4$ is a factor of v(x).

$$\begin{array}{rrrrl}
x-4\,\big) & x^3 & +\,3x^2 & -\,36x & +\,32\;\big(\,x^2+7x-8 \\
 & x^3 & -\,4x^2 & & \\
 & - & + & & \\
\hline
 & & 7x^2 & -\,36x & +\,32 \\
 & & 7x^2 & -\,28x & \\
 & & - & + & \\
\hline
 & & & -8x & +\,32 \\
 & & & -8x & +\,32 \\
 & & & + & - \\
\hline
 & & & & 0
\end{array}$$

$$\begin{aligned}
\text{So, } x^3 + 3x^2 - 36x + 32 &= (x - 4)(x^2 + 7x - 8) \\
&= (x - 4)(x^2 + 8x - x - 8) \\
&= (x - 4)\{x(x + 8) - 1(x + 8)\} \\
&= (x - 4)(x + 8)(x - 1)
\end{aligned}$$

So, the missing measures are $(x + 8)$ and $(x - 1)$

EXERCISE # 1

1. If $f(x) = (2x^3 + x^2 + bx - 6)$ leaves a remainder 36 when divided by $(x - 3)$, find the value of b.

2. Find the coefficient of x^3 in the polynomial $5 + 2x + 3x^2 - 7x^3$.

3. Find the value of $P(x) = x^2 - 7x + 12$ at $x = 3$.

4. Find the number of terms in a polynomial of degree 5.

5. Find the degree of the polynomial $4x^4 + 0x^3 + 0x^5 + 5x + 7$.

6. Find the degree of the zero polynomial.

7. Find the zero of the zero polynomial.

8. Find the one of the zeroes of the polynomial $2x^2 + 7x - 4$.

9. Find the zeroes of the polynomial $p(x) = x(x - 1)(x - 2)$.

10. If $p(x) = x^2 - 2\sqrt{2}x + 1$ then find value of $p(2\sqrt{2})$.

11. If $p(x) = x + 3$, then find $p(x) + p(-x)$.

12. If the polynomial $x^3 + 3x^2 + 3x + 1$ is divided by $x + 1$, then find remainder.

13. If $x^{51} + 51$ is divided by $x + 1$, then find remainder.

14. Find the value of k for which $x - 1$ is a factor of the polynomial $4x^3 + 3x^2 - 4x + k$.

15. Find the factors of $2x^2 - 3x - 2$.

16. Find the factors of $x^3 - 2x^2 - 13x - 10$.

17. If $x^2 + kx + 6 = (x + 2)(x + 3)$ for all x, then find the value of k.

18. Find the value of a, if $(x - a)$ is a factor of $x^3 - ax^2 + x + 2$.

19. When $x^3 + 3x^2 - kx + 4$ is divided by $x - 2$, the remainder is k. Find the value of the constant k.

EXERCISE # 1 ANSWER KEY

1.	−7	2.	−7	3.	0	4.	6 terms
5.	4	6.	Not defined	7.	Any real number		
8.	$\frac{1}{2}$	9.	0, 1, 2	10.	1	11.	6
12.	0	13.	50	14.	−3	15.	$(2x + 1)(x - 2)$
16.	$(x + 1)(x + 2)(x - 5)$			17.	5	18.	−2
19.	8						

EXERCISE # 2

Very short answer type questions

1. Which of the following expressions are polynomial ?

(i) $11x + 1$
(ii) $7x^2 - 5x + \sqrt{5}$
(iii) $t^3 - 2t + 1$
(iv) $x^2 - \frac{1}{x^2}$
(v) $\sqrt{y} + 5y - 1$
(vi) $z^{11} - 5z^7 + \frac{1}{4}$

2. Write the coefficient of x^3 in each of the following :

(i) $3x^3 - 3x + 2$
(ii) $14x^3 - 2x^3 + 5x - 7x^2$
(iii) $\sqrt{2}x^2 + 1$
(iv) $\frac{3}{4}x^3 + 2x - 3$

3. Write the degree of each of the following polynomials

(i) $3x^2 - 4x + 2$
(ii) $7x^3 + 2x^2 + x$
(iii) $5 - x^2$
(iv) $1 + 2x + 3x^2 - 11x^4$

4. Classify the following as linear, quadratic and cubic polynomials :

(i) $x^3 - 4$
(ii) $x^2 + 1$
(iii) $5x^2 - 3x + \sqrt{7}$
(iv) $1 + 5x$
(v) $4r^3$

5. Find the zeroes of each of the following polynomials :

(i) $p(x) = x - 4$
(ii) $g(x) = 2x + 1$
(iii) $p(x) = (x + 1)(x + 2)$
(iv) $p(x) = (x - 1)(x - 2)(x - 3)$
(v) $p(x) = 7x^2$
(vi) $p(x) = rx + s,\ r \neq 0$

Short answer type questions

6. Verify whether the following are zeroes of the polynomial indicated against them :

(i) $p(x) = 5x - 1,\quad x = \frac{1}{5}$
(ii) $p(x) = (x - 2)(x - 5),\quad x = 2, 5$
(iii) $s(x) = x^2,\quad x = 0, 1$
(iv) $p(x) = 3x^2 - 1,\quad x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$
(v) $g(x) = 5x^2 + 7x,\quad x = 0, -\frac{7}{5}$

7. Show that -1 is a zero of the polynomial $2x^3 - x^2 + x + 4$

8. Show that 1 is not a zero of the polynomial $4x^4 - 3x^3 + 2x^2 - 5x + 1$

9. Use remainder theorem to find remainder when p(x) is divided by q(x) in the following questions :

(i) $p(x) = 2x^2 - 5x + 7,\ q(x) = x - 1$
(ii) $p(x) = x^9 - 5x^4 + 1,\ q(x) = x + 1$
(iii) $p(x) = 4x^3 - 12x^2 + 11x - 5,\ q(x) = x - \frac{1}{2}$
(iv) $p(x) = x^4 + x^3 + x^2 - 5x + 1,\ q(x) = x + 1$

10. Use factor theorem to verify in each of the following that q(x) is a factor of p(x)

(i) $p(x) = 3x^2 - 5x + 2,\ q(x) = 3x - 2$
(ii) $p(x) = x^4 - x^3 + x - 1, q(x) = x + 1$
(iii) $p(x) = x^5 - x^4 - 4x^2 - 2x + 4,\ q(x) = x - 2$

11. Find the value of k if $(x - 2)$ is a factor of $2x^3 - 6x^2 + 5x + k$

12. For what value of m is $2x^3 + mx^2 + 11x + m + 3$ exactly divisible by $(2x - 1)$?

13. Using factor theorem, show that $a - b$ is a factor of $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

14. If one of the factors of $x^2 + x - 20$ is $(x + 5)$, find other factor.

Long answer type questions

15. Let A and B are the remainders when the polynomial $y^3 + 2y^2 - 5ay - 7$ and $y^3 + ay^2 - 12y + 6$ are divided by $y + 1$ and $y - 2$ respectively. If $2A + B = 6$, find the value of a.

16. With out actual division, prove that $a^4 + 2a^3 - 2a^2 + 2a - 3$ is exactly divisible by $a^2 + 2a - 3$.

17. If $(x + 1)$ and $(x - 1)$ are the factors of $mx^3 + x^2 - 2x + n$, find the value of m and n

18. If k and 2k are zeros of $f(x) = x^3 + 4x^2 + 9kx - 90$, find k and all three zeros of f(x).

19. The perimeter of a right triangle is 24 centimeters. Three times the length of the longer leg minus two times the length of the shorter leg exceeds the hypotenuse by 2 centimeters. What are the lengths of all three sides?

20. What number should be subtracted from $2x^3 - 5x^2 + 5x$ so that the resulting polynomial has a factor $2x - 3$?

21. If $2x^3 + ax^2 + bx - 2$ has a factor $(x + 2)$ and leaves a remainder 7 when divided by $2x-3$, find the values of a and b. With these values of a and b, factorise the given polynomial completely.

22. What number should be added to $3x^2 + 7x$ so that when the resulting polynomial is divided by $x-3$ the remainder is 2 ?

23. The polynomials $ax^3 + 3x^2 - 13$ and $2x^3 - 5x + a$ are divided by $x + 2$. If the remainder is same in each case, find the value of a.

24. If $x^3 + ax^2 - x + b$ has $(x - 2)$ as a factor and leaves a remainder 3 when divided by $(x - 3)$, find a and b.

25. If $(x-2)$ is a factor of $2x^3 - x^2 - px - 2$,

(i) find the value of p.

(ii) factorize the above expression completely.

26. (a) Find a if $3y - 5$ divides $3y^2 - ay + 10$ completely.

(b) What number should be subtracted from $x^2 + x + 1$ so that the resulting polynomial has a factor $x - 2$?

27. If $4x^2 - 1$ is a factor of $12x^3 + px^2 + qx - 1$ then find p and q.

28. Find a factor of $x^3 + 6x^2 + 11x + 6$ by factor theorem. Hence, factorize it completely.

29. The polynomial $px^2 - 7x^2 + qx + 3$ is divisible by $x + 1$. When it is divided by $2x - 1$, the remainder is $\frac{-15}{8}$.

Find the polynomial and factorize it completely.

30. The polynomials $ax^3 - 7x^2 + 7x - 2$ and $x^3 - 2ax^2 + 8x - 8$ when divided by $x - 2$ leave the same remainder. Find the value of a.

EXERCISE # 2 ANSWER KEY

Very short answer type questions

1. (i), (ii), (iii), (vi)

2. (i) 3, (ii) –2, (iii) 0, (iv) $\frac{3}{4}$

3. (i) 2, (ii) 3, (iii), 2, (iv) 4

4. (i) Cubic, (ii) Quadratic, (iii) Quadratic, (iv) Linear, (v) Cubic

5. (i) 4, (ii) –1/2, (iii) –1, –2, (iv) 1, 2, 3, (v) 0, (vi) –s/r

Short answer type questions

6. (i) Yes, (ii) Yes, both, (iii) x = 0, x = 1 is not zero, (iv) $x = -\frac{1}{\sqrt{3}}$, $x = \frac{2}{\sqrt{3}}$ is not zero, (v) Yes, both

9. (i) 4, (ii) –5, (iii) –2, (iv) 7

11. –2

12. –7

14. (x – 4)

Long answer type questions

15. 2

17. m = 2, n = – 1

18. k = – 3, zeros are –3, –6 & 5

19. 8cm, 6cm, 10cm

20. [3]

21. a = 3, b = –3 and (x + 2)(x – 1)(2x + 1)

22. –46

23. 5/9

24. a= –3, b = 6

25. 5, $2x^2 + 3x + 1$

26. a=11, k = 7

27. p = 4, q = –3

28. (x + 1) (x + 2) (x + 3)

29. $3x^2 - 7x^2 - 7x + 3$, (x + 1) (x – 3) (3x – 1)

30. a = 2